LEAST-SQUARES METHODS FOR OPTIMAL CONTROL

PAVEL BOCHEV[1]

Department of Mathematics, University of Texas at Arlington, Box 19408, Arlington, TX 76019-0408
E-mail: bochev@utamat.uta.edu

Key words and phrases: Least-squares principles, Navier-Stokes equations, boundary control, finite elements.

1. INTRODUCTION

Optimal control and optimal shape design problems for the Navier-Stokes equations arise in many important practical applications, such as design of optimal profiles [7], drag minimization [9], [11], and heating and cooling [12], among others. Typically, optimal control problems for the Navier-Stokes equations combine Lagrange multiplier techniques to enforce the constraints and to derive an optimality system (see, e.g., [11]-[12]), with mixed Galerkin discretization for the state equations. Resulting methods are well-studied theoretically, for example, an abstract framework that can be used for the analyses of such optimal control methods has been suggested in [13]. However, the use of Lagrange multipliers and mixed Galerkin discretizations is associated with some complications in the numerical computations which can reduce the overall efficiency and robustness of corresponding algorithms. For example, resulting discrete problems are in general indefinite. Similarly, it is now well-understood that stability of mixed discretizations does not allow one to choose independently the approximation spaces for the velocity and the pressure, and that these spaces are subject to a restrictive stability condition known as the inf-sup (or LBB) condition; see [8]. One possibility to remedy these difficulties is to consider optimal control methods in which the Navier-Stokes constraint is treated by augmented Lagrangian techniques; see [6]. Nevertheless, the use of mixed Galerkin discretization in the method of [6] still requires approximation by finite element spaces that are subject to the inf-sup condition.

In this paper we consider another alternative which involves the use of least-squares variational principles. Such principles have been successfully used for the approximation of the incompressible, steady-state Navier-Stokes equations, see, e.g., [2]-[3], [14], and [15], among others. The main idea of least-squares methods is to consider minimization of appropriately defined quadratic functionals, i.e., least-squares variational principles correspond to minimization, rather than to a saddle-point optimization problem. This allows one to circumvent the inf-sup condition and to achieve stable discretizations using, e.g., equal order interpolation for all unknowns. Likewise, in a neighborhood of the minimizer, the Hessian matrix is guaranteed to be symmetric and positive definite. As a result, using, e.g., Newton linearization combined with continuation with respect to the Reynolds number, one can devise methods that will encounter only symmetric and positive definite linear systems in the solution process.

[1]Supported in part by REP Grant of the University of Texas

In this paper the least-squares approach will be extended to optimal control problems. Our goal is to develop a finite element method for boundary control of the steady-state, incompressible Navier-Stokes equations in which least-squares principles are used to derive the first-order optimality system. As a result, discrete problems associated with this system now correspond to positive definite algebraic problems. It appears that with the exception of [1], where an optimal design problem arising in semiconductor applications has been treated by a least-squares method, this is the first application of least-squares ideas for optimal flow control.

The rest of this paper is organized as follows. In §2 we introduce the optimal control problem along with two particular model problems. The least-squares approach for this problem is developed in §3. In the same section we show how the least-squares method specializes to the model problems. Section 4 concludes the paper with presentation of some computational results obtained with the least-squares optimal control method.

Throughout this paper Ω will denote a bounded, open region in $\mathcal{R}^2$ with Lipschitz continuous boundary Γ. We use the standard Sobolev space notation $H^s(\Omega)$ for the set of all functions defined on Ω such that all their derivatives of order up to s are square integrable. Similarly, for the norm and the inner product on $H^s(\Omega)$ we use the standard notations $\|\cdot\|_s$ and $(\cdot,\cdot)_s$, respectively. As usual, when $s=0$ we use $L^2(\Omega)$ instead of $H^0(\Omega)$. With $\mathbf{H}^s(\Omega)$ we shall denote the corresponding vector Sobolev space, for example, $\mathbf{H}^1(\Omega)=H^1(\Omega)\times H^1(\Omega)$. $H^1_0(\Omega)$ will denote the subspace of all functions in $H^1(\Omega)$ that vanish on Γ, and $L^2_0(\Omega)$ will denote the subspace of all zero-mean functions in $L^2(\Omega)$. Lastly, $H^{-1}(\Omega)$ will be used to denote the dual space of $H^1_0(\Omega)$.

2. MODEL OPTIMAL CONTROL PROBLEMS

The Navier-Stokes equations for steady, viscous, incompressible flow can be written in the form (see [10])

$$
\begin{aligned}
-\nu\left(\mathbf{grad}\,\mathbf{u}+\mathbf{grad}\,\mathbf{u}^T\right)+\mathbf{u}\cdot\mathbf{grad}\,\mathbf{u}+\mathbf{grad}\,p &= \mathbf{f} \ \text{ in } \Omega && (2.1)\\
\operatorname{div}\mathbf{u} &= 0 \ \text{ in } \Omega && (2.2)\\
\mathbf{u} &= \mathbf{g} \ \text{ on } \Gamma, && (2.3)
\end{aligned}
$$

where $\mathbf{u}$, p and $\mathbf{f}$, denote velocity field, pressure, and a given body force, ν is the kinematic viscosity, and $\mathbf{g}$ is given function such that $\int_\Gamma \mathbf{n}\cdot\mathbf{g}\,d\Gamma=0$. Uniqueness of the solutions to (2.1)-(2.3) requires that $p\in L^2_0(\Omega)$.

To state the optimal control problem for (2.1)-(2.3) let $\mathcal{J}(\mathbf{u},p,\mathbf{g})$ be a given cost functional where $\mathbf{g}$ denotes the (boundary) control. We define the admissibility set as

$$
\mathcal{U}=\left\{(\mathbf{u},p,\mathbf{g})\in\mathbf{H}^1(\Omega)\times L^2_0(\Omega)\times H^{1/2}(\Gamma);\ \mathbf{g}=\mathbf{g}_0+\sum_{i=1}^n l_i\mathbf{g}_i \ \big|\ (\mathbf{u},p,\mathbf{g}) \text{ satisfy (2.1)-(2.3)}\right\} \qquad (2.4)
$$

where $\mathbf{g}_i\in H^{1/2}(\Gamma)$; $l_i\in\Lambda_i$, for $i=0,1,\dots,n$; and $\Lambda_i\in\mathcal{R}$ are closed intervals. Then, the problems we shall consider are given by

seek $(\mathbf{u},p,\mathbf{g})\in\mathcal{U}$ *such that* $\mathcal{J}(\mathbf{u},p,\mathbf{g})\le\mathcal{J}(\mathbf{v},q,\mathbf{h})$ *for all* $(\mathbf{v},q,\mathbf{h})\in\mathcal{U}$.

This problem will be used to state the least-squares finite element method for optimal boundary control. To illustrate the method and to present the numerical results it will be specialized for two particular model problems that are introduced below.

Both of our model optimal control problems will be considered in the context of the fictitious Driven Cavity flow, i.e., we assume that $\Omega = [0,1]^2$ and that there is no body force. We use Γ_L, Γ_R and Γ_B, Γ_T to denote the left and the right, and the bottom and the top surfaces of Ω, respectively. The first problem we shall consider has been suggested in [6] and can be stated as follows.

> Given the bottom velocity $\mathbf{u}|_{\Gamma_B} = \mathbf{u}_B$, find the top velocity $\mathbf{u}|_{\Gamma_T} = \mathbf{u}_T$ such that the separation of the flow occurs at a desired horizontal line location Γ_S, see Fig.1.

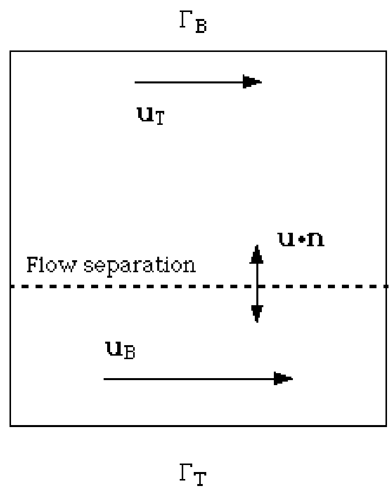

Fig. 1. Flow separation for the Driven Cavity

As a result, the control objective for this problem is to minimize the following cost functional

$$\mathcal{J}(\mathbf{u}, p, \mathbf{g}) = \int_{\Gamma_S} |\mathbf{u}_2|^2 \, d\Gamma \tag{2.5}$$

using controls of the form $\mathbf{g} = \mathbf{g}_0 + \mathbf{u}_T \mathbf{g}_1$ where

$$\mathbf{g}_0 = \begin{cases} (\mathbf{u}_B, 0) & \text{on } \Gamma_B \\ (0,0) & \text{otherwise} \end{cases} \quad ; \quad \mathbf{g}_1 = \begin{cases} (1,0) & \text{on } \Gamma_T \\ (0,0) & \text{otherwise} \end{cases} ,$$

see [6]. Note that for this problem we have that $\int_\Gamma \mathbf{n} \cdot \mathbf{g}_i \, d\Gamma = 0$ for $i = 0, 1$.

For the second model problem we consider minimization of the flow vorticity in a subdomain $\Omega_1 \subset \Omega$ (see Fig. 2), i.e., we consider minimization of a cost functional given by

$$\mathcal{J}(\mathbf{u}, p, \mathbf{g}) = \int_{\Omega_1} |\mathbf{curl}\,\mathbf{u}|^2 \, d\Omega \,. \tag{2.6}$$

We let $\Omega_1 = [0.75, 1.00] \times [0, 0.25]$ and use $\hat{\Gamma}$ to denote the boundary of Ω_1. For this problem we consider controls of the form $\mathbf{g} = \mathbf{g}_0 + l_1\mathbf{g}_1 + l_2\mathbf{g}_2$ where

$$\mathbf{g}_0 = \begin{cases} (1,0) & \text{on } \Gamma_T \\ (0,0) & \text{otherwise} \end{cases} ; \quad \mathbf{g}_1 = \begin{cases} (0,1) & \text{on } \hat{\Gamma}_B \\ (0,0) & \text{otherwise} \end{cases} ; \quad \mathbf{g}_2 = \begin{cases} (1,0) & \text{on } \hat{\Gamma}_R \\ (0,0) & \text{otherwise} \end{cases}$$

The choice of the control here corresponds to suction or injection through the bottom and the right portions of the boundary of Ω_1, i.e., in this case we have that $\int_\Gamma \mathbf{n} \cdot \mathbf{g}_i \, d\Gamma \neq 0$ for $i = 1, 2$.

We note that the functions $\mathbf{g}_i$ used to define the control $\mathbf{g}$ in both model problems are not in $H^{1/2}(\Gamma)$. Thus, as in [6], in what follows we assume that these functions are replaced by $C^\infty(\Gamma)$ approximations.

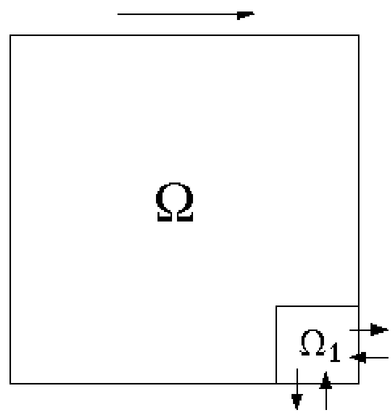

Fig. 2. Vorticity minimization over a subdomain

3. LEAST-SQUARES OPTIMAL CONTROL METHODS

In this section we develop the least-squares approach for the optimal control problem of §2, and then specialize it for the two model problems. Formulation of the least-squares method will involve three steps: decomposition of the state equations (2.1)-(2.3) into an equivalent first-order system followed by definition of a quadratic cost functional and an associated least-squares variational principle, and lastly, discretization of the resulting variational problem. The need to include a decomposition step stems from practical considerations. Since least-squares functionals will involve weighted L^2-norms of the residuals of the state equations, the use of a first-order system allows discretization by means of standard finite element spaces.

To recast the incompressible Navier-Stokes equations as a first-order system one usually introduces velocity derivatives (or their linear combinations) as new dependent variables. There are several possible ways to do this. For example, one can choose the new variables according to $\underline{\mathbf{U}} = \mathbf{grad}\,\mathbf{u}$. This leads to a "velocity flux-velocity-pressure" first-order system; see [5]. Two other natural choices are to use the symmetric or the skew-symmetric parts of the velocity gradient, i.e, to introduce $1/2(\mathbf{grad}\,\mathbf{u} + \mathbf{grad}\,\mathbf{u}^T)$, or $1/2(\mathbf{grad}\,\mathbf{u} - \mathbf{grad}\,\mathbf{u}^T)$ as new dependent variables. The first choice leads to a "stress-velocity-pressure" first-order system, whereas the second choice corresponds to a "velocity-vorticity-pressure" first-order system, see [2]-[3], [14], [15]. Among the three systems the velocity-vorticity-pressure one has been used most often in the context of least-squares methods, and resulting finite element algorithms are well-documented and studied. For this reason here we shall stick to this form of the Navier-Stokes equations.

To define velocity-vorticity-pressure form of (2.1)-(2.3) we recall the two-dimensional curl operators **curl** and curl given by

$$\mathbf{curl}\,\phi = \begin{pmatrix} \phi_y \\ -\phi_x \end{pmatrix} \qquad \text{and} \qquad \mathrm{curl}\,\mathbf{u} = u_{2x} - u_{1y}\,,$$

respectively. To avoid multiplicity of notations we introduce the "vector" products $\phi \times \mathbf{u}$ and $\mathbf{v} \times \mathbf{u}$, where ϕ is a scalar function, and $\mathbf{u}$, $\mathbf{v}$ are vectors in $\mathcal{R}^2$, by embedding ϕ, $\mathbf{u}$ and $\mathbf{v}$ into the three-dimensional vectors $(0, 0, \phi)$, $(u_1, u_2, 0)$ and $(v_1, v_2, 0)$ respectively. Then, $\mathbf{curl}\,\phi = \nabla \times \phi$ and $\mathrm{curl}\,\mathbf{u} = \nabla \times \mathbf{u}$. Furthermore, we agree to use **curl** in both cases and to denote the result as a vector. Then, we introduce the vorticity $\boldsymbol{\omega} = \mathbf{curl}\,\mathbf{u}$ as a new dependent variable. In view of the vector identities

$$\mathbf{curl}\,\mathbf{curl}\,\mathbf{u} = -\triangle\mathbf{u} + \mathbf{grad}\,\mathrm{div}\,\mathbf{u}\,,$$

$$\mathbf{u} \cdot \mathbf{grad}\,\mathbf{u} = \frac{1}{2}\mathbf{grad}\,|\mathbf{u}|^2 - \mathbf{u} \times \mathbf{curl}\,\mathbf{u}$$

one can rewrite the momentum equation (2.1) as $\nu\mathbf{curl}\,\boldsymbol{\omega} + \boldsymbol{\omega} \times \mathbf{u} + \mathbf{grad}\,r = \mathbf{f}$, where $r = p + 1/2|\mathbf{u}|^2$ denotes the total head. As a result, we obtain the following first-order *velocity-vorticity-pressure* Navier-Stokes equations

$$\begin{aligned} \nu\mathbf{curl}\,\boldsymbol{\omega} + \boldsymbol{\omega} \times \mathbf{u} + \mathbf{grad}\,r &= \mathbf{f} \ \text{ in } \Omega & (3.1)\\ \mathbf{curl}\,\mathbf{u} - \boldsymbol{\omega} &= 0 \ \text{ in } \Omega & (3.2)\\ \mathrm{div}\,\mathbf{u} &= 0 \ \text{ in } \Omega\,, & (3.3) \end{aligned}$$

along with the boundary condition (2.3).

Following [6] we transform the boundary control problem into a distributed control problem as follows. For $i = 0, \dots, n$ we let $(\boldsymbol{\omega}_i, \mathbf{u}_i, r_i)$ denote solutions of the Stokes problem

$$\begin{aligned} \nu\mathbf{curl}\,\boldsymbol{\omega}_i + \mathbf{grad}\,r_i &= \mathbf{0} \ \text{ in } \Omega & (3.4)\\ \mathbf{curl}\,\mathbf{u}_i - \boldsymbol{\omega}_i &= 0 \ \text{ in } \Omega & (3.5)\\ \mathrm{div}\,\mathbf{u}_i &= 0 \ \text{ in } \Omega & (3.6)\\ \mathbf{u}_i &= \mathbf{g}_i \ \text{ on } \Gamma\,. & (3.7) \end{aligned}$$

Then we set $\mathbf{u} = \hat{\mathbf{u}} + \mathbf{u}_0 + \sum_{i=1}^n l_i\mathbf{u}_i$; $\boldsymbol{\omega} = \hat{\boldsymbol{\omega}} + \boldsymbol{\omega}_0 + \sum_{i=1}^n l_i\boldsymbol{\omega}_i$; $r = \hat{r} + r_0 + \sum_{i=1}^n l_i r_i$, where $(\hat{\boldsymbol{\omega}}, \hat{\mathbf{u}}, \hat{r}) \in L^2(\Omega) \times \mathbf{H}^1_0(\Omega) \times L^2_0(\Omega)$. As a result, the controls l_i now appear directly in the cost functional $\mathcal{J}$. Furthermore, in view of the new system (3.1)-(3.3) the admissibility set (2.4) is replaced by

$$\mathcal{U} = \left\{ (\hat{\boldsymbol{\omega}}, \hat{\mathbf{u}}, \hat{r}, \mathbf{l}) \in L^2(\Omega) \times \mathbf{H}^1_0(\Omega) \times L^2_0(\Omega) \times \prod_{i=1}^n \Lambda_i \ | \ (\boldsymbol{\omega}, \mathbf{u}, r) \text{ satisfy (3.1)-(3.3), (2.3)} \right\}. \quad (3.8)$$

Then, the optimal control problem for the first-order system (3.1)-(3.3), and (2.3) can be stated as the following constrained minimization problem:

seek $(\hat{\boldsymbol{\omega}}, \hat{\mathbf{u}}, \hat{r}, \mathbf{l}) \in \mathcal{U}$ *such that* $\mathcal{J}(\hat{\boldsymbol{\omega}}, \hat{\mathbf{u}}, \hat{r}, \mathbf{l}) \leq \mathcal{J}(\hat{\boldsymbol{\xi}}, \hat{\mathbf{v}}, \hat{q}, \mathbf{k})$ *for all* $(\hat{\boldsymbol{\xi}}, \hat{\mathbf{v}}, \hat{q}, \mathbf{k}) \in \mathcal{U}$.

With this control problem we associate a quadratic least-squares functional given by

$$\begin{aligned} J(\boldsymbol{\omega}, \mathbf{u}, r, \mathbf{l}) &= \alpha_1 \|\nu \mathbf{curl}\, \boldsymbol{\omega} + \boldsymbol{\omega} \times \mathbf{u} + \mathbf{grad}\, r - \mathbf{f}\|_{-1}^2 \\ &+ \alpha_2 \|\mathbf{curl}\, \mathbf{u} - \boldsymbol{\omega}\|_0^2 + \alpha_3 \|\operatorname{div} \mathbf{u}\|_0^2 + \alpha_4 \mathcal{J}(\hat{\boldsymbol{\omega}}, \hat{\mathbf{u}}, \hat{r}, \mathbf{l}), \end{aligned} \tag{3.9}$$

and a least-squares minimization principle given by

seek $(\hat{\boldsymbol{\omega}}, \hat{\mathbf{u}}, \hat{r}, \mathbf{l}) \in \mathbf{X} \times \Lambda$ *such that* $J(\boldsymbol{\omega}, \mathbf{u}, r, \mathbf{l}) \leq J(\boldsymbol{\xi}, \mathbf{v}, q, \mathbf{k})$ *for all* $(\hat{\boldsymbol{\xi}}, \hat{\mathbf{v}}, \hat{q}, \mathbf{k}) \in \mathbf{X} \times \Lambda$,

where $\mathbf{X} = L^2(\Omega) \times \mathbf{H}_0^1(\Omega) \times L_0^2(\Omega)$ and $\Lambda = \prod_{i=1}^n \Lambda_i$. Note that in contrast to the original control problem the least-squares optimal control formulation above corresponds to an unconstrained minimization problem.

A first-order optimality system for the least-squares control problem is provided by the Euler-Lagrange equation for the functional (3.9). This equation constitutes a nonlinear variational problem which we write symbolically as

seek $(\hat{\boldsymbol{\omega}}, \hat{\mathbf{u}}, \hat{r}, \mathbf{l}) \in \mathbf{X} \times \Lambda$ *such that*

$$Q((\boldsymbol{\omega}, \mathbf{u}, r, \mathbf{l}); (\boldsymbol{\xi}, \mathbf{v}, q, \mathbf{k})) = 0 \quad \forall (\hat{\boldsymbol{\xi}}, \hat{\mathbf{v}}, \hat{q}, \mathbf{k}) \in \mathbf{X} \times \Lambda. \tag{3.10}$$

The form of $Q(\cdot;\cdot)$ in (3.10) depends on the particular cost functional $\mathcal{J}$ used in the control problem.

The last step in the development of the least-squares control method deals with the discretization of the problem (3.10). This step involves selection of the finite element spaces, approximation of the functions $(\boldsymbol{\omega}_i, \mathbf{u}_i, r_i)$, and replacement of $Q(\cdot;\cdot)$ by a computable discrete equivalent. The need for such a replacement stems from the fact that in addition to the usual L^2 inner products the form $Q(\cdot;\cdot)$ in (3.10) also involves the inner product of the *negative order* Sobolev space $H^{-1}(\Omega)$. This inner product is not computable and must be replaced by a computable discrete equivalent.

Let us first consider the choice of the discretization. For this purpose here we shall use biquadratic finite element spaces. More precisely, let $\mathcal{T}_h$ denote a regular triangulation of the domain Ω into rectangles and let Q_2 denote the set of all functions which are polynomials of degree less than or equal to 2 in each of the coordinate directions. The parameter h above can be identified with some measure of the elements in $\mathcal{T}_h$, e.g., their diameter. Then we define the space

$$Q^h = \{u^h \in C^0(\Omega) \,|\, u^h|_\square \in Q_2(\square), \quad \square \in \mathcal{T}_h\} \tag{3.11}$$

of *biquadratic* finite elements, and the corresponding vector space

$$\mathbf{Q}^h = \{\mathbf{u}^h \in Q^h \times Q^h \,|\, \mathbf{u}^h = \mathbf{0} \quad \text{on } \Gamma\}.$$

The discrete counterpart of the minimization space $\mathbf{X} \times \Lambda$ that appears in the least-squares control problem is, therefore, given by

$$\mathbf{X}^h \times \Lambda = Q^h \times \mathbf{Q}^h \times Q^h \cap L_0^2(\Omega) \times \prod_{i=1}^n \Lambda_i. \tag{3.12}$$

Next, the functions $\boldsymbol{\omega}_i$, $\mathbf{u}_i$ and r_i are replaced by finite element approximations denoted by $\boldsymbol{\omega}_i^h$, $\mathbf{u}_i^h$ and r_i^h, respectively. These approximations can be computed using a least-squares method for (3.4)-(3.6), see [2]. Finally, using a scaling argument, one can infer that for finite element functions the H^{-1}-norm that appears in (3.9) can be replaced by the weighted L^2-norm $h\|\cdot\|_0$. (a more sophisticated approach, suggested in [4], uses discrete negative norms defined by means of preconditioners for the Laplace's equation). As a result, for finite element functions we consider the minimization of the following least-squares functional

$$\begin{aligned} J^h(\boldsymbol{\omega}^h, \mathbf{u}^h, r^h, \mathbf{l}) &= \alpha_1 h^2 \|\nu \mathbf{curl}\, \boldsymbol{\omega}^h + \boldsymbol{\omega}^h \times \mathbf{u}^h + \mathbf{grad}\, r^h - \mathbf{f}\|_0^2 \\ &+ \alpha_2 \|\mathbf{curl}\, \mathbf{u}^h - \boldsymbol{\omega}^h\|_0^2 + \alpha_3 \|\mathrm{div}\,^h \mathbf{u}\|_0^2 + \alpha_4 \mathcal{J}(\hat{\boldsymbol{\omega}}^h, \hat{\mathbf{u}}^h, \hat{r}^h, \mathbf{l})\,, \end{aligned} \tag{3.13}$$

where $\mathbf{u}^h = \hat{\mathbf{u}}^h + \mathbf{u}_0^h + \sum_{i=1}^n l_i \mathbf{u}_i^h$, $\boldsymbol{\omega}^h = \hat{\boldsymbol{\omega}}^h + \boldsymbol{\omega}_0^h + \sum_{i=1}^n l_i \boldsymbol{\omega}_i^h$, and $r^h = \hat{r}^h + r_0^h + \sum_{i=1}^n l_i r_i^h$. A discrete first-order optimality system is then provided by the Euler-Lagrange equation for (3.13), which we write again symbolically as

seek $(\hat{\boldsymbol{\omega}}^h, \hat{\mathbf{u}}^h, \hat{r}^h, \mathbf{l}) \in \mathbf{X}^h \times \Lambda$ *such that*

$$B^h((\boldsymbol{\omega}^h, \mathbf{u}^h, r^h, \mathbf{l}); (\boldsymbol{\xi}^h, \mathbf{v}^h, q^h, \mathbf{k})) = 0 \quad \forall (\hat{\boldsymbol{\xi}}^h, \hat{\mathbf{v}}^h, \hat{q}^h, \mathbf{k}) \in \mathbf{X}^h \times \Lambda\,, \tag{3.14}$$

where $\mathbf{v}^h = \hat{\mathbf{v}}^h + \mathbf{u}_0^h + \sum_{i=1}^n l_i \mathbf{u}_i^h$, $\boldsymbol{\xi}^h = \hat{\boldsymbol{\xi}}^h + \boldsymbol{\omega}_0^h + \sum_{i=1}^n l_i \boldsymbol{\omega}_i^h$, and $q^h = \hat{r}^h + q_0^h + \sum_{i=1}^n l_i r_i^h$.

Next we consider how (3.14) can be specialized to our model problems. For brevity we provide the details only for the first model problem, the differences in (3.14) for the second one are not essential. For this problem we may assume without loss of generality that $\Lambda = \Lambda_1 = [0, 1]$. Then the first-order discrete optimality system (3.14) specializes to the following set of equations

seek $(\hat{\boldsymbol{\omega}}^h, \hat{\mathbf{u}}^h, \hat{r}^h, l) \in \mathbf{X}^h \times \Lambda$ *such that*

$$\begin{aligned} \alpha_1 h^2 \int_\Omega (\nu \mathbf{curl}\, \hat{\boldsymbol{\omega}}^h + \mathbf{grad}\, \hat{r}^h + \boldsymbol{\omega}^h \times \mathbf{u}^h) \cdot (\nu \mathbf{curl}\, \hat{\boldsymbol{\xi}}^h + \mathbf{grad}\, \hat{q}^h + \hat{\boldsymbol{\xi}}^h \times \mathbf{u}^h + \boldsymbol{\omega}^h \times \hat{\mathbf{v}}) dx & \\ + \quad \alpha_2 \int_\Omega \mathrm{div}\, \hat{\mathbf{u}}^h \mathrm{div}\, \hat{\mathbf{v}}^h dx & \\ + \quad \alpha_3 \int_\Omega (\mathbf{curl}\, \hat{\mathbf{u}}^h - \hat{\boldsymbol{\omega}}^h)(\mathbf{curl}\, \hat{\mathbf{v}}^h - \hat{\boldsymbol{\xi}}^h) dx + & \\ + \quad \alpha_4 \int_{\Gamma_S} \mathbf{u}_2 \hat{\mathbf{v}}_2 d\Gamma = 0 \qquad \text{for all } (\boldsymbol{\xi}^h, \mathbf{v}^h, q^h) \in \mathbf{X}^h & \\ 0 = a_3 l^3 + a_2 l^2 + a_1 l + a_0\,. & \end{aligned} \tag{3.15}$$

The coefficients a_i above are given by

$$\begin{aligned} a_3 &= 2\alpha_1 \|\boldsymbol{\omega}_1^h \times \mathbf{u}_1^h\|_0^2 \\ a_2 &= 3\alpha_1 \int_\Omega (\boldsymbol{\omega}_1^h \times \mathbf{u}_1^h) \cdot (\boldsymbol{\omega}_1 \times (\mathbf{u}^h + \mathbf{u}_0^h) + (\boldsymbol{\omega}^h + \boldsymbol{\omega}_0^h) \times \mathbf{u}_1^h) dx \\ a_1 &= \alpha_1 \|\boldsymbol{\omega}_1 \times (\mathbf{u}^h + \mathbf{u}_0^h) + (\boldsymbol{\omega}^h + \boldsymbol{\omega}_0^h) \times \mathbf{u}_1^h\|_0^2 \end{aligned}$$

$$
\begin{aligned}
&+ \quad 2\alpha_1 \int_\Omega (\boldsymbol{\omega}_1^h \times \mathbf{u}_1^h) \cdot (\nu \mathbf{curl}\, \boldsymbol{\omega}^h + \mathbf{grad}\, r^h + (\boldsymbol{\omega}^h + \boldsymbol{\omega}_0^h) \times (\mathbf{u}^h + \mathbf{u}_0^h)) dx \\
&+ \quad \alpha_4 \int_{\Gamma_S} (\mathbf{u}_{2\,1}^h)^2 d\Gamma \\
a_0 \quad &= \quad 2\alpha_1 \int_\Omega (\boldsymbol{\omega}_1 \times (\mathbf{u}^h + \mathbf{u}_0^h) + (\boldsymbol{\omega}^h + \boldsymbol{\omega}_0^h) \times \mathbf{u}_1^h) \cdot \\
& \qquad (\nu \mathbf{curl}\, \boldsymbol{\omega}^h + \mathbf{grad}\, r^h + (\boldsymbol{\omega}^h + \boldsymbol{\omega}_0^h) \times (\mathbf{u}^h + \mathbf{u}_0^h)) dx \\
&+ \quad \alpha_4 \int_{\Gamma_S} (\mathbf{u}_2^h + \mathbf{u}_{2\,0}^h) \mathbf{u}_{2\,1}^h d\Gamma \,.
\end{aligned}
$$

It is not difficult to see that once a basis for the space Q^h is selected, (3.15) corresponds to a nonlinear system of algebraic equations that must be solved in an iterative manner. Although the explicit form of (3.15) is quite formidable this system also has some quite valuable computational properties. First, in a neighborhood of a minimizer for (3.13) the Jacobian of (3.15) is necessarily symmetric and positive definite. Second, since the attraction ball of Newton's method is nontrivial, one is assured the existence of an initial approximation for the Newton's method, such that both the Newton's method will converge, and the linearized system will be symmetric and positive definite. As a result, the linearized system can be solved using efficient and robust iterative methods, such as conjugate gradients, i.e., the method can be implemented without assembly of the discretization matrix, even at the element level. In particular, here we have implemented the Newton's method for the solution of (3.15) using conjugate gradients with Jacobi preconditioning

4. NUMERICAL RESULTS

This section presents numerical results obtained with the least-squares method for optimal control. All computations were carried on using biquadratic finite elements defined with respect to a uniform triangulation of the unit square Ω into rectangles. In particular we use triangulations of 19 by 19 rectangular elements, i.e., the number of grid points in each coordinate direction equals 39. The nonlinear algebraic system is solved using Newton's method with tolerance set to $0.5/10^5$.

In the first experiment we consider the separation problem for the Driven Cavity flow. For this experiment weights have been chosen according to $\alpha_1 = \alpha_2 = \alpha_3 = 1$; $\alpha_4 = 1/h$. The separation line Γ_S is chosen to be a horizontal line through the geometric center of the cavity, and $\mathbf{u}_B$ is set equal to 0.1 As a result, the expected optimal value for $\mathbf{u}_T$ is 0.1. After three iterations Newton's method converged within the prescribed tolerance and the computed approximation for $\mathbf{u}_T$ was found to be 0.0989. On Fig. 3 computed optimal flow is compared with a Driven Cavity flow corresponding to the boundary condition $\mathbf{u}_\Gamma = (0.1, 0)$ on Γ_T and Γ_B, and zero otherwise. As one can see from the plots presented in Fig. 3, the flow computed using the least-squares optimal control method is in a very good agreement with the expected flow.

In our second experiment we consider minimization of the vorticity over a prescribed subdomain Ω_1 using suction or injection through the bottom and the right portions of the boundary of Ω_1. For this example all weights have been chosen equal to 1. We have carried computations for values of the Reynolds number up to $Re = 612.6$. For higher values of Re Newton's method diverged, which hints at the possibility that the optimal control problem may not have a unique solution, or a solution at all. This is consistent with the analyses of [6], where existence of optimal solutions for a similar problem has been established under the assumption that the Reynolds number is relatively low. Our numerical results are presented on Fig. 4. which contains contour plots of the vorticity variable for increasing values of Re. Directions of the arrows on each plot indicate the combinations of suction and/or injection that correspond to the optimal solution computed with the least-squares method. We see that these combinations vary with the Reynolds number. For example, for low and high values of Re optimal solutions are obtained using injection along the vertical boundary of Ω_1 and suction along the horizontal boundary of this domain, whereas for intermediate values of Re the opposite holds true.

References

[1] BEDIVAN D. & FIX G. J., Least-squares methods for optimal shape design problems, *Computers Math. Applic.*, **30**,2, 17-25 (1995).

[2] BOCHEV P., Analysis of least-squares methods for the Navier-Stokes equations, to appear in *SIAM J. Num. Anal.*

[3] BOCHEV P. & GUNZBURGER M., Accuracy of least-squares methods for the Navier-Stokes equations, *Comput. & Fluids*, **22**, 4/5, 549-563, (1993).

[4] BRAMBLE J. H., LAZAROV R. D. &. PASCIAK J. E, A least squares approach based on a discrete minus one inner product for first order systems, Technical Report 94-32, Mathematical Science Institute, Cornell University, 1994.

[5] CAI Z., MANTEUFFEL T.A., & MCCORMICK S. F., First-order system least-squares for the Stokes equations, with application to linear elasticity, to appear in *SIAM Numer. Anal.*.

[6] DESAI M., & ITO K., Optimal Control of Navier-Stokes Equations, *SIAM J. Control and Optim.*, **32**, 1428-1446(1994).

[7] FERNANDEZ CARA E., Optimal design in fluid mechanics. The finite element approach, in: *Proc. IFIP WG.72 Working Conference*, Santiago de Compostela 1987, edited by A. Bermudez, Springer 120-131 (1989).

[8] GIRAULT V. & RAVIART P.-A., *Finite Element Methods for Navier-Stokes Equations*, Springer, Berlin, 1986.

[9] GLOWINSKI R. & PIRONNEAU O., Toward the computation of minimum drag profiles in viscous laminar flow, *Appl. Math. Model.*, **1**, 58-66 (1976).

[10] GUNZBURGER M., *Finite Element Methods for Viscous Incompressible Flows*, Academic, Boston, 1989.

[11] GUNZBURGER M., HOU L., & SVOBODNY T., Boundary velocity control of incompressible flow with an application to drag reduction, *SIAM J. Control and Optim.*, **30**, 167-181 (1992).

[12] GUNZBURGER M., HOU L., & SVOBODNY T., *Heating and cooling control of temperature distributions along boundaries of flow domains* ICAM Report 91-10-01, ICAM, Virginia Tech (1991).

[13] GUNZBURGER M. & HOU L., Finite-dimensional approximation of a class of constrained nonlinear optimal control problems, *SIAM J. Control Optim.* **34/3**, 1001-1043 (1996).

[14] JIANG B.-N., LIN T., & POVINELLI L., A least-squares finite element method for 3D incompressible Navier-Stokes equations, AIAA Report 93-0338, (1993).

[15] LEFEBVRE D., PERAIRE J., & MORGAN K., Least-squares finite element solution of compressible and incompressible flows, *Int. J. Num. Meth. Heat Fluid Flow* **2**, 99-113 (1992).

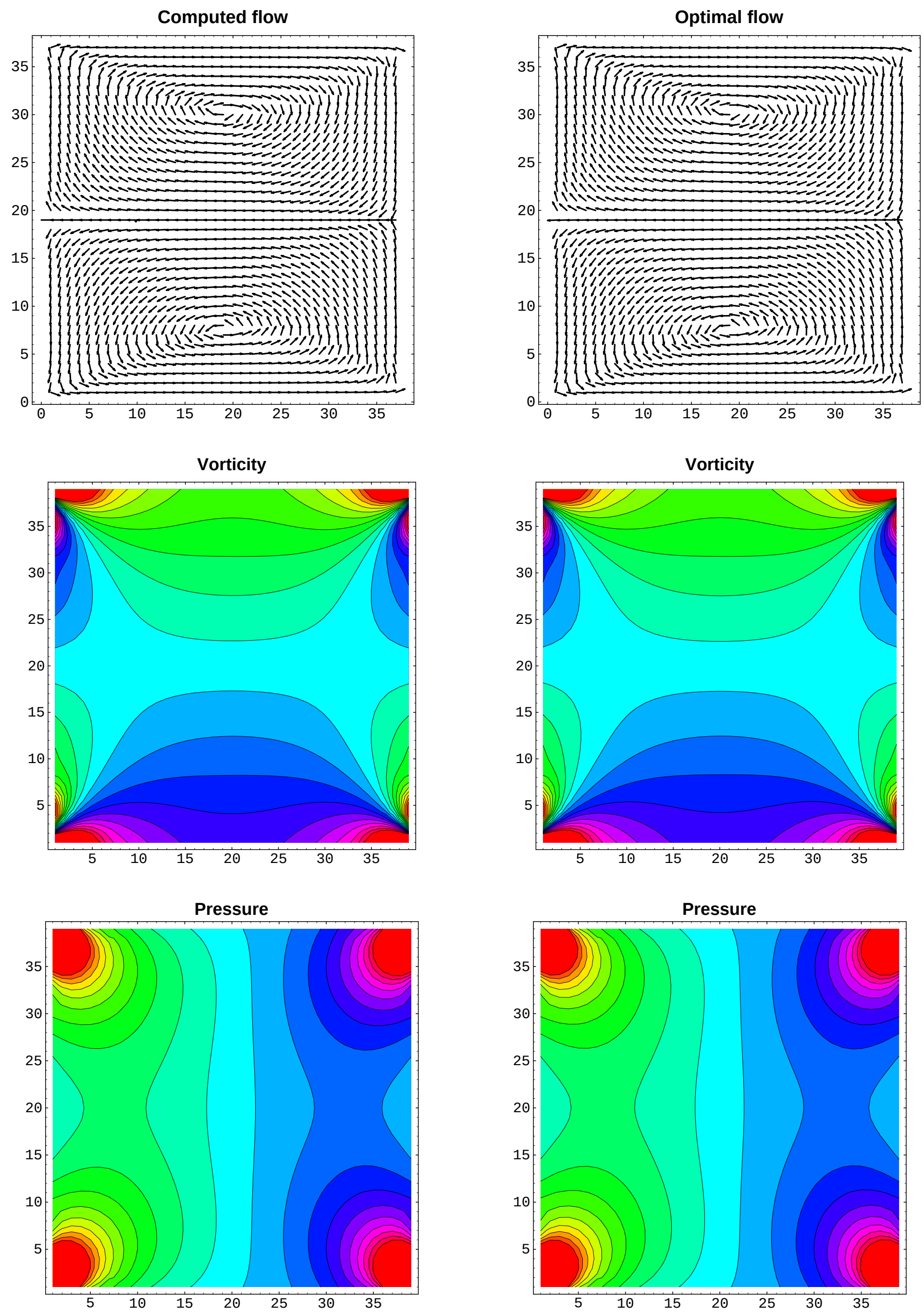

Fig. 3. Velocity field, vorticity and pressure contours for computed and optimal flows: $Re = 10$, $\mathbf{u}_B = 0.1$, separation at $y = 0.5$.

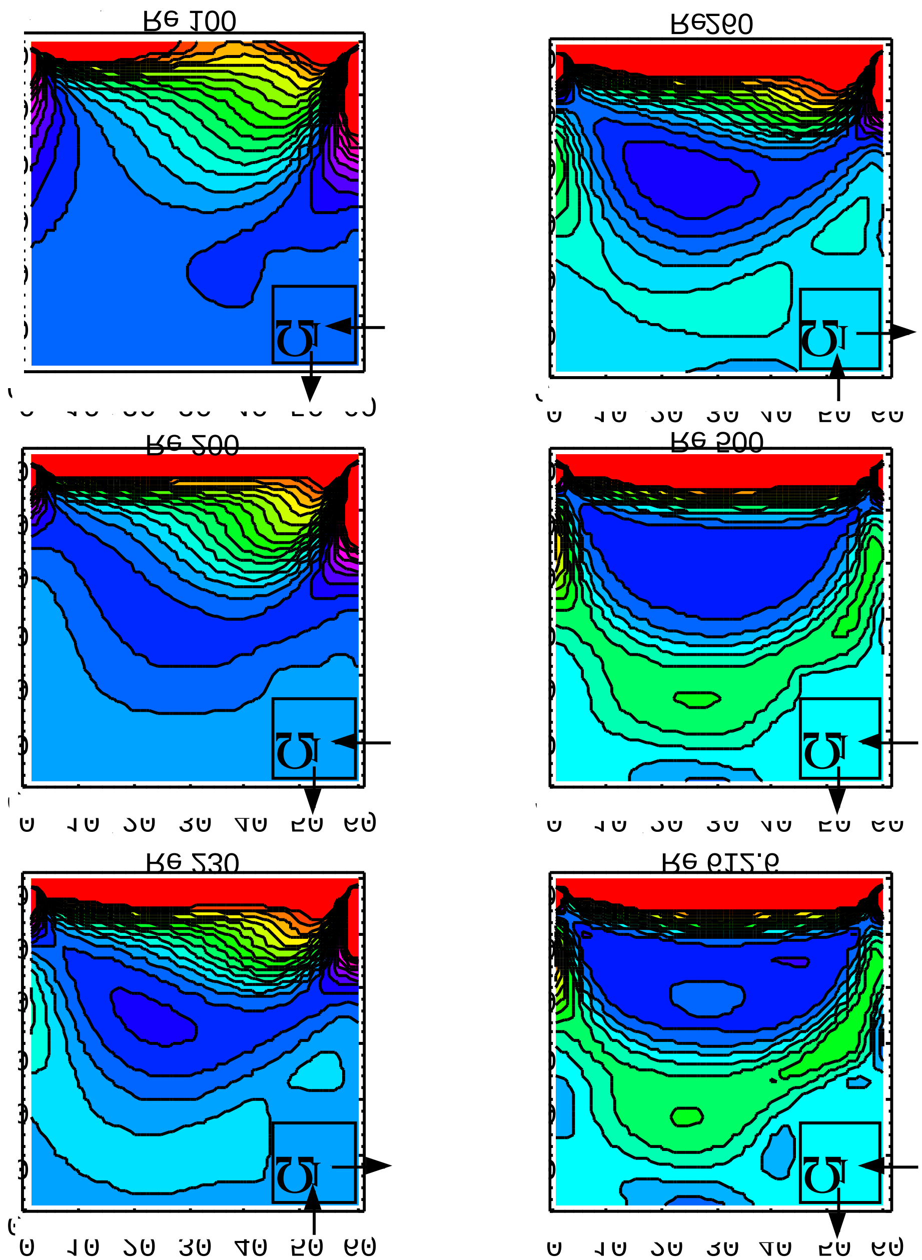

Fig. 4. Vorticity contours for different values of Re for the second model problem